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On the inviscid instability of certain two-dimensional vortex-type flows

By ALFONS MICHALKE AND ADALBERT TIMME

Deutsche Versuchsanstalt für Luft- und Raumfahrt e. V.
Institut für Turbulenzforschung, Berlin

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As a contribution to the breakdown phenomenon of vortices in a two-dimensional free boundary layer, this paper deals with the question whether a single cylindrical (i.e. two-dimensional) vortex can become unstable. For this reason a single vortex, as it occurs in a free boundary layer, is approximated by an axisymmetrical vortex model. The inviscid stability theory of rotating fluids is then applied to this vortex model. By general stability criteria it was found that a vortex consisting of vorticity of one sign only is stable according to the Rayleigh criterion, but, if the vorticity has an extremum value outside the axis, may become unstable with respect to cylindrical disturbances. Furthermore, stability calculations for three special types of vortex were performed. It was found that they were more unstable with respect to cylindrical disturbances than to three-dimensional ones.

1. Introduction

In this paper an attempt is made by means of inviscid stability theory to look for a reason for the breakdown of vortices in a disturbed two-dimensional free boundary layer which occurs in jets with large cores.

It is known that the laminar free boundary layer is unstable under classical inviscid linearized stability theory because of the inflexion point in its velocity profile (cf. Rayleigh 1880). Experimental investigations among others by Wille (1952), Sato (1960), Schade & Michalke (1962), Michalke & Wille (1966), Freymuth (1966) have shown that for large Reynolds numbers this instability occurs in fact and is nearly independent of the Reynolds number. It can be described by means of the inviscid linearized stability theory of spatially growing disturbances as was shown by Michalke (1965*b*). Furthermore, it was found in the experiments that the disturbed free boundary layer rolls up into vortices. These vortices, however are certainly not of the type described by potential theory, but as real vortices they may be defined by postulating the existence of a local concentration of vorticity. The vortex model suggests itself, because, for example, the slipping motion of two consecutive vortices, which can often be observed in experiments, can well be explained by this model. An attempt to explain the formation of vortices in free boundary layers was made by Schade & Michalke (1962), Michalke (1965*a*) and Michalke & Freymuth (1966). Freymuth (1966) found that even the formation of vortices is not essentially influenced by viscosity so that the rolling-up process should also be described by the inviscid equation of motion.

The experiments show that further downstream the vortices finally break down and the free boundary layer becomes turbulent. Thus the question arises as to why the vortices break down. Domm (1956) supposed that the vortices, because of their growth and by the possible coalescence of two consecutive vortices to a bigger one, reach a critical state such that they become unstable due to centrifugal forces. Timme (1957), who investigated the vortices in a Kármán vortex street, introduced a 'Reynolds number' which was assumed to be proportional to the ratio of the circulation of the vortices to the kinematic viscosity of the fluid. He supposed that the vortices break down if this 'Reynolds number' has reached a critical value. Wehrmann & Wille (1958) applied this concept to the vortices in jet boundary layer, while Fabian (1960) tried to verify it by hot-wire measurements in the free boundary layer of an axisymmetric jet. This hypothesis, however, seems to be unjustified, since recently Freymuth (1966) has found that viscosity plays no important role in the vortex breakdown in free shear layers. He stated that the mutual induction of the vortices may be responsible for their three-dimensional decay. On the other hand, a further reason for vortex breakdown may be that the vorticity distribution in a single vortex moving downstream with constant velocity becomes unstable, which finally leads to turbulence. In the more complicated case of the leading-edge vortices of a delta-wing, the theoretical results of an inviscid stability theory developed by Ludwieg (1960, 1964) seem to be well confirmed by the experimental results of Hummel (1964, 1965). Another theory of vortex breakdown has been developed by Benjamin (1962). Nevertheless, for the vortices in jet boundary layers the question of vortex instability seems to be unanswered. Thus we will treat this problem in what follows by means of inviscid stability theory.

In order to investigate the stability of a single vortex, one has to know its undisturbed vorticity distribution. If, as found in the experiments, the formation of vortices in a two-dimensional free boundary layer escaping out of a nozzle is not essentially influenced by viscosity, the convection of vorticity must be described by the inviscid vorticity equation. This means, however, that the vorticity is fixed at any fluid particle and remains constant as the particle moves downstream along its path. At the nozzle the flow is nearly stationary. If smoke is introduced into that region of the flow containing most of the vorticity, the fluid particles which pass by through the same initial points at different time will be marked by that smoke. The particles form streaklines which can be observed. On the other hand, since at the initial points the flow is assumed to be stationary, the vorticity along each streakline must be constant in time. Therefore each streakline must be identical with a line of constant vorticity according to the inviscid vorticity equation.

Under these assumptions the smoke distribution observed downstream in the shear layer will be identical with the vorticity distribution. It is, however, known from smoke experiments (cf. Freymuth 1966) and shown schematically in figure 1, that the free boundary layer which contains the vorticity produced at the nozzle wall becomes thicker, with a simultaneous folding at the point where a vortex is formed. This folding-process observed in the experiments is equivalent to a local concentration of vorticity and agrees with the streakline pattern found

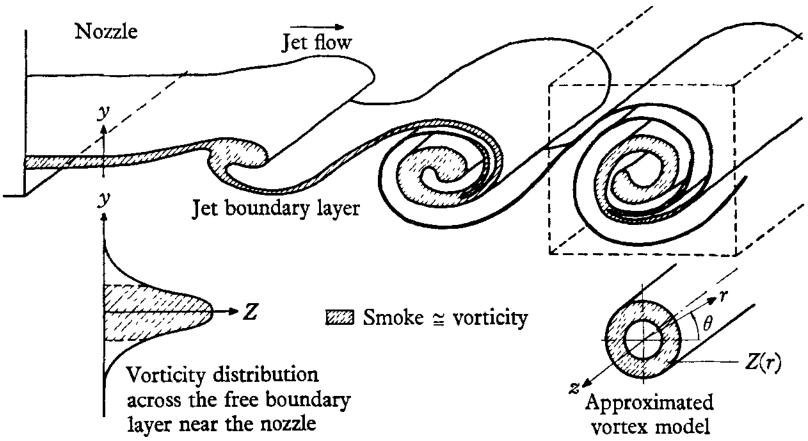

FIGURE 1. The rolling-up process of a free boundary layer and the approximated vortex model.

theoretically (cf. Michalke & Freymuth 1966). Further downstream the streak-lines roll up. Finally, in a fully developed vortex, most of the vorticity must be concentrated on a nearly circular band as shown on the right in figure 1. These vortices move downstream with a constant phase velocity which is nearly half of the jet velocity.

For the stability analysis a frame of reference is used which moves with the vortex. It is assumed that in this frame of reference the vortex is stationary, because the influence of the viscosity is neglected. In order to facilitate the analysis, we furthermore assume an inviscid single vortex with a simplified vorticity distribution, i.e. axisymmetrical as shown at the bottom right of figure 1. Then the vorticity Z depends only on the distance r from the vortex axis, that is $Z = Z(r)$, and the vortex induces only a circumferential velocity component $V = V(r)$. For this approximate vortex model the stability theory of rotating fluids is applicable. As mentioned above we assume further that vortex breakdown is an inviscid phenomenon. Hence we restrict ourselves to inviscid stability theory.

Since Rayleigh (1917) the necessary and sufficient inviscid stability criterion for inviscid rotating fluids has been known. According to this, a flow is stable if the square of the circulation increases for increasing radius. If we use the relation between the circulation and the circumferential velocity $V(r)$ and the vorticity $Z(r)$ respectively, this criterion is equivalent to the condition that the product VZ must always be positive for stability. In our case this condition is always satisfied. Since the vorticity Z contained in the free boundary layer is of one sign only, the velocity V has the same sign. Hence VZ is greater than zero for every radius, and the vortex must be stable according to the Rayleigh criterion.

Little attention, however, has sometimes been paid to the fact that the Rayleigh criterion was derived for axisymmetrical disturbances only. In his book Chandrasekhar (1961) tried to prove that the Rayleigh criterion is also valid for arbitrary disturbances, but evidently he was in error as was shown by Howard & Gupta (1962). There they discussed the general disturbance equation for a flow having a circumferential as well as an axial velocity component.

After Rayleigh, many papers concerning the stability of rotating fluids were published, but mostly they were restricted to Couette flow—the most famous paper was given by Taylor (1923)—or to the small-gap problem as treated, for example, by Ludwieg (1960, 1964). Since we are interested in the stability of an infinitely extended vortex-type flow with arbitrary vorticity distribution $Z = Z(r)$, we shall give in §2 the disturbance equation. In §3 general aspects of the disturbance equation will be discussed, while in §4 examples of unstable vortex type flows which are stable according to the Rayleigh criterion will be investigated.

2. The disturbance equation for inviscid rotating fluids

Any inviscid rotating flow with a velocity component V in the circumferential direction only is a solution of the Euler equation of motion in a (r, θ, z)-cylindrical co-ordinate system, if

$$V = V(r). \tag{2.1}$$

Then the corresponding vorticity has a component $Z(r)$ in z-direction only with

$$Z(r) = r^{-1}(rV)' = V' + V/r, \tag{2.2}$$

where primes denote differentiation with respect to r.

In order to study the instability of such a basic flow, we superimpose small three-dimensional disturbance. If we only consider normal modes, we have the disturbances

$$\left.\begin{matrix} u_{r1}(r,\theta,z,t) \\ u_{\theta 1}(r,\theta,z,t) \\ u_{z1}(r,\theta,z,t) \\ p_1(r,\theta,z,t) \end{matrix}\right\} = \mathscr{R} \left\{\begin{matrix} u_r(r) \\ u_\theta(r) \\ u_z(r) \\ p(r) \end{matrix}\right\} \exp\left[i(m\theta + kz - \beta t)\right], \tag{2.3}$$

where m should be an integer for physical reasons, k a real constant (the wave-number in z-direction) and $\beta = \beta_r + i\beta_i$ is complex. β_r is the disturbance cyclic frequency and β_i the temporal growth rate.

Inserting (2.1) and (2.3) into the Euler equation of motion and the continuity equation neglecting higher order terms in the disturbances we obtain a system of four equations for the amplitude functions $u_r(r)$, $u_\theta(r)$, $u_z(r)$ and $p(r)$ of the disturbances (2.3). These equations can be reduced to a single one for $u_r(r)$.

If we introduce

$$\phi = ru_r, \tag{2.4}$$

we obtain finally

$$[rf\phi']' - [r^{-1} + F]\phi = 0, \tag{2.5}$$

with

$$f = [m^2 + k^2r^2]^{-1}, \tag{2.6}$$

$$F = \frac{m(Zf)'}{m(V/r) - \beta} - \frac{2k^2VZf}{[m(V/r)-\beta]^2}. \tag{2.7}$$

The connexion of $\phi(r)$ with the amplitude functions of the original disturbances (2.3) is given by (2.4) and

$$u_\theta = if[m\phi' + (k^2/\sigma)rZ\phi], \tag{2.8}$$

$$u_z = ikf[r\phi' - (m/\sigma)Z\phi], \tag{2.9}$$

$$p = if[\sigma r\phi' - mZ\phi], \tag{2.10}$$

where

$$\sigma(r) = m(V/r) - \beta. \tag{2.11}$$

The analogous disturbance equation to (2.5)—except for a misprint—was given by Howard & Gupta (1962) for a flow which includes also an axial basic velocity $W(r)$.

The boundary conditions to be satisfied by the disturbance are determined by the condition that for fixed cylindrical walls at $r = r_1$ and $r = r_2$ the normal velocity u_r must vanish, that is

$$\frac{\phi(r_1)}{r_1} = \frac{\phi(r_2)}{r_2} = 0. \tag{2.12}$$

If the origin belongs to the flow, it is sufficient to require that $u_r(0)$ is finite, that is

$$\phi(0) = 0. \tag{2.13}$$

The homogeneous disturbance equation (2.5) together with the homogeneous boundary conditions (2.12) define an eigenvalue problem. For fixed values (m, k) solutions of (2.5) are sought which satisfy (2.12). This is only possible for certain eigenvalues $\beta = \beta_r + i\beta_i$ which will depend on (m, k). Since (2.5) is not changed if we replace k by $-k$ or m by $-m$ and β by $-\beta$ we can restrict ourselves to positive values of (m, k).

3. General features of the disturbance equation

From the disturbance equation (2.5) and the boundary conditions (2.12) and (2.13) respectively some general criteria can be derived without solving the differential equation for a special flow.

If we suppose $\beta_i \neq 0$ and write

$$\phi(r) = r\sigma^{1-\mu}\chi(r) \tag{3.1}$$

in (2.5) and multiply the equation by $r\sigma^{1-\mu}$ we obtain the differential equation for the function $\chi(r)$,

$$[r^3\sigma^{2(1-\mu)}f\chi']' - F_\mu \,.\, r^2 \,.\, \sigma^{2(1-\mu)}\chi = 0, \tag{3.2}$$

where

$$F_\mu = \frac{1}{r}[1-(rf)'] + \frac{m}{\sigma}\left[\mu(Zf)' + 2(1-\mu)\frac{V}{r}f'\right] - \frac{f}{\sigma^2}\left[2k^2VZ - \mu(1-\mu)rm^2\left(\frac{V}{r}\right)'^2\right]. \tag{3.3}$$

Let us first discuss (3.2) for the special case $\mu = 0$ and $k = 0$. Then (3.2) yields

$$[r^3\sigma^2\chi']' - (m^2-1)r\sigma^2\chi = 0. \tag{3.4}$$

We see that for $m = 1$ a solution is $\chi =$ constant and, therefore,

$$\phi(r) \sim V - \beta r \tag{3.5}$$

is a solution of (2.5). Thus for a flow in $0 \leqslant r < \infty$ with $V(0) = 0$ and

$$\lim_{r\to\infty} V(r) = c_1 + c_2 r$$

the boundary conditions (2.13) and (2.12) for $m = 1$ are satisfied, if

$$\beta_r = c_2, \quad \beta_i = 0. \tag{3.6}$$

Therefore (3.5) is a neutral solution of (2.5), although it is not believed that (3.5) and (3.6) are the limit case $\beta_i \to 0$ of an eigenfunction with $\beta_i \neq 0$.

Furthermore, let us multiply equation (3.2) by the conjugate complex function $\bar{\chi}$ and integrate over (r_1, r_2). Then we get by means of partial integration and the boundary conditions at (r_1, r_2)

$$\int_{r_1}^{r_2} r^3 f \sigma^{2(1-\mu)} |\chi'|^2 dr + \int_{r_1}^{r_2} r^2 \sigma^{2(1-\mu)} |\chi|^2 F_\mu \, dr = 0. \tag{3.7}$$

From the integral relation (3.7) one can obtain necessary conditions for instability of an inviscid rotating flow.

Taking first $\mu = 0$ one can easily deduce from the imaginary part of (3.7) that a necessary condition for instability ($\beta_i \neq 0$) is for $V \geqslant 0$ that

$$\frac{\beta_r}{m} < \max\left(\frac{V}{r}\right). \tag{3.8}$$

For the special case of a cylinder-symmetric disturbance ($k = 0$) we find

$$\min\left(\frac{V}{r}\right) < \frac{\beta_r}{m} < \max\left(\frac{V}{r}\right). \tag{3.9}$$

For the other special case of an axisymmetric disturbance ($m = 0$) the Rayleigh criterion can be derived from (3.7), namely that a necessary condition for instability ($\beta_i \neq 0$) is

$$\left.\begin{array}{l} \beta_r = 0, \\ VZ < 0 \quad \text{somewhere in} \quad r_1 < r < r_2. \end{array}\right\} \tag{3.10}$$

Taking $\mu = 1$ a further criterion found by Howard & Gupta (1962) is obtained from the imaginary part of (3.7): a necessary condition for instability ($\beta_i \neq 0$) is that the term

$$m(Zf)' - 4k^2\, VZf |\sigma|^{-2} \left(m\frac{V}{r} - \beta_r\right) \tag{3.11}$$

must change sign in $r_1 < r < r_2$. For the special case of a cylinder-symmetric disturbance ($k = 0$), (3.11) implies $Z' = 0$ somewhere in $r_1 < r < r_2$. That means that the vorticity $Z(r)$ must have an extremum value inside the flow region. This condition for instability is equivalent to the inflexion point criterion for parallel flows.

If we put finally $\mu = \frac{1}{2}$ we obtain from the imaginary part of (3.7) the necessary condition for instability ($\beta_i \neq 0$) that

$$8k^2 VZ - m^2 r (V/r)'^2 < 0 \tag{3.12}$$

somewhere in $r_1 < r < r_2$. This criterion was also found by Howard & Gupta (1962). For $VZ > 0$ and $m \neq 0$ (3.12) suggests that the three-dimensional disturbance with $k \neq 0$ is more stable than the corresponding cylinder-symmetric disturbance alone ($k = 0$). This would mean that in this case a behaviour must be expected which is analogous to that for plane parallel flow as stated by Squire (1933). In fact, if we transform the co-ordinates (r, θ, z) to a new system (y, x, z) by means of

$$r = R+y; \quad \theta = x/R; \quad z = z; \tag{3.13}$$

where R denotes a radius in the vicinity of which the vorticity is concentrated, then using

$$V(r) = U(y); \quad m = \alpha R; \quad u_r(r) = v(y); \tag{3.14}$$

we obtain from the disturbance equation (2.5)

$$\left\{\left(1+\frac{y}{R}\right)\tilde{f}\left[\left(1+\frac{y}{R}\right)v\right]'\right\}' - \left(1+\frac{y}{R}\right)\left[\frac{1}{1+y/R}+\frac{\alpha(Z\tilde{f})'}{\sigma}-\frac{2k^2}{R}\frac{VZ\tilde{f}}{\sigma^2}\right]v = 0, \tag{3.15}$$

where

$$\left.\begin{aligned} \tilde{f} &= \left[\alpha^2+k^2\left(1+\frac{y}{R}\right)^2\right]^{-1}, \\ \sigma &= \frac{\alpha U}{1+(y/R)} - \beta, \\ Z &= \frac{1}{1+(y/R)}\left[\left(1+\frac{y}{R}\right)U\right]'. \end{aligned}\right\} \tag{3.16}$$

If we now let $R \to \infty$ and have $y/R \ll 1$, then (3.15) tends to

$$v'' - \left[\alpha^2+k^2+\frac{U''}{U-\beta/\alpha}\right]v = 0. \tag{3.17}$$

This is the three-dimensional inviscid disturbance equation for plane parallel flow for which the theorem of Squire is valid.

4. Examples of vortex instability

In this chapter the inviscid instability of certain vortex-type flows is investigated. We will restrict ourselves to inviscid, infinitely extended flows for which the Rayleigh criterion for stability is satisfied, i.e. $VZ \geqslant 0$ in $0 \leqslant r \leqslant \infty$. This implies that the vortex consists of a vorticity concentration of one sign only. Let us assume that for $VZ \geqslant 0$ and a fixed value of m the disturbance is more stable for $k \neq 0$ than for $k = 0$ as mentioned above. Then instability can only occur if the vorticity $Z(r)$ has an extremum value outside the vortex axis, i.e. for $r > 0$. Therefore a stationary Hamel–Oseen-type vortex with

$$Z(r) = \text{const.}\, e^{-a^2r^2}, \tag{4.1}$$

where a is any constant, must be stable and also the Rankine vortex with

$$\left.\begin{aligned} Z(r) &= \text{const.} \quad \text{for} \quad 0 \leqslant r < r_0, \\ Z(r) &= 0 \quad \text{for} \quad r_0 < r. \end{aligned}\right\} \tag{4.2}$$

The simplest vortex type which should be unstable seems to be a vortex which is given by a cylindrical vortex sheet.

4.1. *The instability of the cylindrical vortex sheet*

The velocity field induced by a cylindrical vortex sheet having a radius R is given by

$$\left.\begin{aligned} v_1(r) &= 0 \quad \text{for} \quad 0 \leqslant r < R, \\ v_2(r) &= V_0 R/r \quad \text{for} \quad R < r, \end{aligned}\right\} \tag{4.3}$$

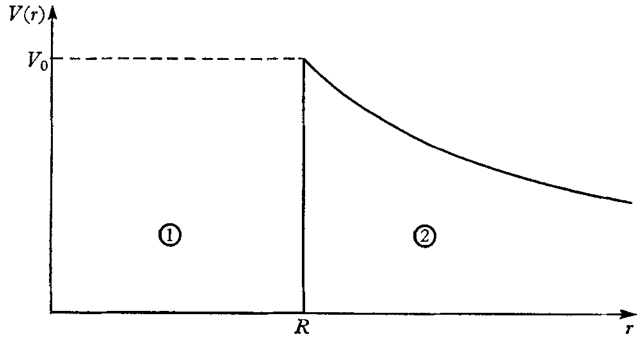

FIGURE 2. The velocity distribution induced by a cylindrical vortex sheet.

and is shown in figure 2. Inside the two regions 1 and 2 the vorticity is $Z \equiv 0$ and hence by (2.19) $F \equiv 0$. Thus the disturbance equation (2.18) becomes

$$\left[\frac{r}{m^2+k^2r^2}\phi'\right]' - \frac{1}{r}\phi = 0. \tag{4.4}$$

The general solution of (4.4) can be obtained by introducing

$$\phi = rw', \tag{4.5}$$

which gives from (4.4)

$$w'' + \frac{1}{r}w' - \left[k^2 + \frac{m^2}{r^2}\right]w = 0. \tag{4.6}$$

This equation is satisfied by the modified Bessel functions. Hence the general solution of (4.4) inside the two regions is given by

$$\phi_{1,2}(r) = A_{1,2}k^{-m}kr\frac{dI_m(kr)}{d(kr)} + B_{1,2}k^m kr\frac{dK_m(kr)}{d\,(kr)}. \tag{4.7}$$

The normalizing factors k^{-m} and k^m are introduced in order to allow $k \to 0$ for $m^2+k^2 \neq 0$ which gives

$$\lim_{k\to 0}\phi_{1,2} = A^*_{1,2},r^m + B^*_{1,2}r^{-m}, \tag{4.8}$$

where

$$A^*_{1,2} = \frac{2^{-m}}{(m-1)\,!}A_{1,2}; \quad B^*_{1,2} = -m\,!\,2^{m-1}B_{1,2}. \tag{4.9}$$

The boundary condition (2.21) at $r = 0$ requires $B_1 = 0$, while the other at $r = \infty$ gives $A_2 = 0$. Thus

$$\left.\begin{aligned} \phi_1(r) &= A_1 k^{-m}\,kr\frac{dI_m(kr)}{d(kr)}, \\ \phi_2(r) &= B_2 k^m\,kr\frac{dK_m(kr)}{d(kr)}. \end{aligned}\right\} \tag{4.10}$$

In order to determine the remaining constants A_1 and B_2, it is convenient to require that

(i) the discontinuity sheet must be constant in time and moves with the fluid,

(ii) the pressure on both sides of the sheet must have the same value. Hence, if the equation of the disturbed discontinuity sheet is given by

$$H(r,\theta,z,t) = r - H_1(\theta,z,t) = 0, \tag{4.11}$$

then the condition to be satisfied is

$$\frac{dH}{dt} = 0 \tag{4.12}$$

or in the linearized form with (2.3)

$$\frac{\partial H_1}{\partial t} + \frac{V}{r}\frac{\partial H_1}{\partial \theta} = u_{r1}. \tag{4.13}$$

In our case we assume

$$H_1(\theta,z,t) = R + \mathscr{R}[C\exp\{i(m\theta + kz - \beta t)\}]. \tag{4.14}$$

Then we find a condition equivalent to (4.13):

$$\lim_{\epsilon\to 0}\{\phi\,[mV-\beta r]^{-1}|_{r=R+\epsilon} - \phi[mV-\beta r]^{-1}|_{r=R-\epsilon}\} = 0. \tag{4.15}$$

The pressure condition (ii) gives from (2.10)

$$\lim_{\epsilon\to 0}\{[mV-\beta r]\,\phi'|_{r=R+\epsilon} - [mV-\beta r]\,\phi'|_{r=R-\delta}\} = 0. \tag{4.16}$$

From (4.15) and (4.10) we obtain

$$A_1 k^{-m}\left[m - \frac{\beta R}{V_0}\right]\kappa I'_m(\kappa) + B_2 k^m \frac{\beta R}{V_0}\kappa K'_m(\kappa) = 0, \tag{4.17}$$

where

$$\kappa = kR \tag{4.18}$$

and from (4.16)

$$A_1 k^{-m}\frac{\beta R}{V_0} I_m(\kappa) + B_2 k^m\left[m - \frac{\beta R}{V_0}\right] K_m(\kappa) = 0, \tag{4.19}$$

where use is made of the differential equation (4.6) satisfied by the modified Bessel functions.

The homogeneous linear system for A_1 and B_2 of (4.17) and (4.19) has non-trivial solutions if its determinant vanishes. Hence we obtain a quadratic equation for the eigenvalues β. Using the Wronskian of the modified Bessel functions

$$I'_m(\kappa)\,K_m(\kappa) - K'_m(\kappa).I_m(\kappa) = \kappa^{-1} \tag{4.20}$$

we find

$$\beta = \frac{V_0}{R} m\left\{\kappa I'_m(\kappa)\,K_m(\kappa) \pm \left(\kappa I'_m(\kappa)\,K_m(\kappa)[\kappa I'_m(\kappa) K_m(\kappa) - 1]\right)^{\frac{1}{2}}\right\}. \tag{4.21}$$

Amplified disturbances ($\beta_i \neq 0$) are only possible, if

$$\kappa I'_m(\kappa)\,K_m(\kappa)\,[\kappa I'_m(\kappa)\,K_m(\kappa) - 1] < 0. \tag{4.22}$$

Since $I'_m(\kappa) \geqslant 0$, $K'_m(\kappa) \leqslant 0$ and $I_m(\kappa) \geqslant 0$, $K_m(\kappa) \geqslant 0$, it follows from (4.20) that (4.22) is always satisfied. Therefore the eigenvalues are

$$\left.\begin{aligned} \beta_r &= \frac{V_0}{R} m\kappa I'_m(\kappa)\,K_m(\kappa), \\ \beta_i &= \frac{V_0}{R} m\left((\kappa I'_m(\kappa)\,K_m(\kappa)\,[1-\kappa I'_m(\kappa)\,K_m(\kappa)]\right)^{\frac{1}{2}}. \end{aligned}\right\} \tag{4.23}$$

Because
$$\lim_{\kappa\to 0} \kappa I'_m(\kappa) K_m(\kappa) = \frac{1}{2}, \tag{4.24}$$
the eigenvalues become for the special case $k = 0$
$$\lim_{\kappa\to 0} \beta_r = \lim_{\kappa\to 0} \beta_i = \frac{V_0}{R}\frac{m}{2}. \tag{4.25}$$
On the other hand, since
$$\lim_{\kappa\to\infty} \kappa I'_m(\kappa) K_m(\kappa) = \frac{1}{2}, \tag{4.26}$$
we have also
$$\lim_{\kappa\to\infty} \beta_r = \lim_{\kappa\to\infty} \beta_i = \frac{V_0}{R}\frac{m}{2}. \tag{4.27}$$
Finally, for fixed κ
$$\lim_{m\to\infty} \kappa I'_m(\kappa) K_m(\kappa) = \frac{1}{2}. \tag{4.28}$$

Thus for large values of m the eigenvalues become independent of k and tend to the solution (4.25) for $k = 0$. For large values of the radius R of the cylindrical vortex sheet the limits of the eigenvalues by (4.26) and (3.14) are
$$\lim_{R\to\infty} \beta_r = \lim_{R\to\infty} \beta_i = V_0 \tfrac{1}{2}\alpha, \tag{4.29}$$
which is known to be the solution for the plane vortex sheet. In figure 3 and figure 4 the influence of the axial wave-number k upon the frequency and the growth rate respectively is shown. We see that the influence of k is small—note the enlarged scales—expecially on the growth rate. Furthermore, for $k \neq 0$, β_i does not exceed the value for $k = 0$. Thus the three-dimensional disturbance is no more unstable than the two-dimensional one. Nevertheless, the cylindrical vortex sheet is always unstable for $m \geqslant 1$.

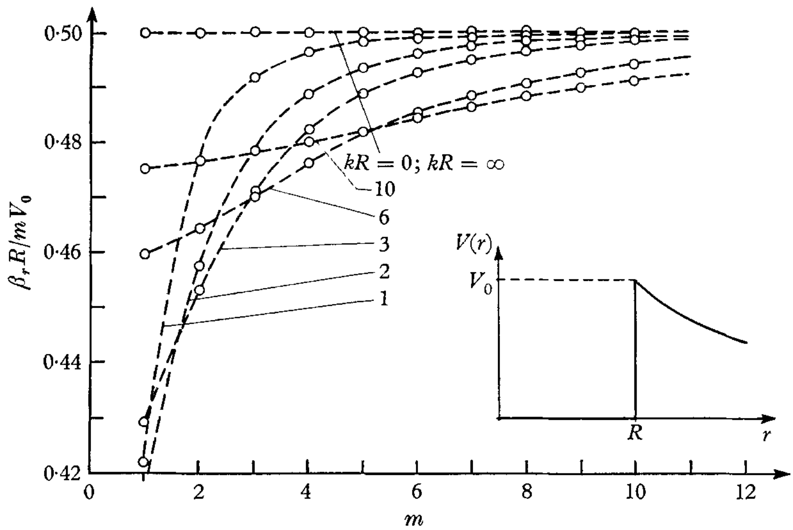

FIGURE 3. The influence of the axial wave-number k on the unstable frequencies.

4.2. *The instability of a vortex with uniform vorticity*

In §4.1 it was found that the infinitely thin cylindrical vortex sheet is always unstable for $m \geqslant 1$. Let us now investigate the case of a vortex consisting of

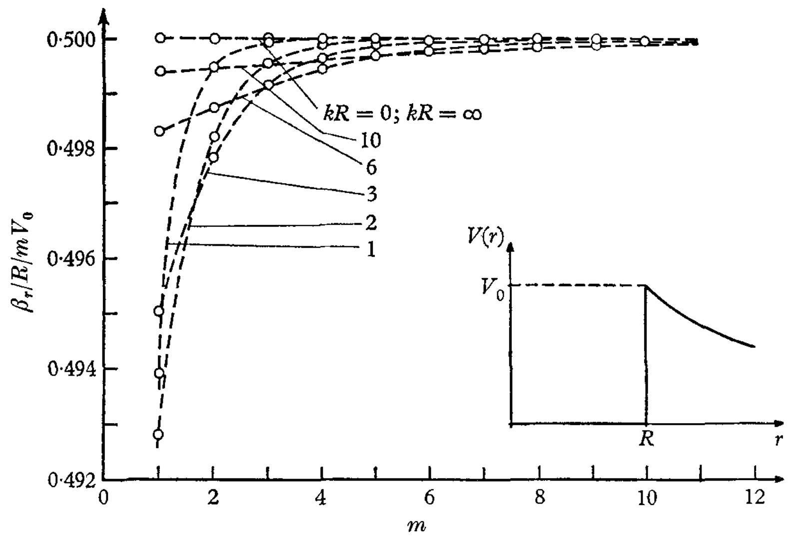

FIGURE 4. The influence of the axial wave-number k on the temporal growth rate.

finitely thick cylindrical vortex sheets. In order to study the influence of the shape of the velocity distribution we assume a vorticity distribution taking constant but different values in each of three regions. Thus we have in a normalized form

$$\left.\begin{aligned} Z_1(r) &= Z_1 \quad \text{in} \quad 0 \leqslant r < \delta, \\ Z_2(r) &= Z_2 \quad \text{in} \quad \delta < r < 1, \\ Z_3(r) &= 0 \quad \text{in} \quad 1 < r < \infty, \end{aligned}\right\} \tag{4.30}$$

with $0 < \delta < 1$. The corresponding velocity distribution is according to (2.2)

$$\left.\begin{aligned} V_1(r) &= 0{\cdot}5\, Z_1 r, \\ V_2(r) &= 0{\cdot}5\, Z_2[r - r^{-1}] + r^{-1}, \\ V_3(r) &= r^{-1}. \end{aligned}\right\} \tag{4.31}$$

Since $V(r)$ is to be a continuous function, it must be required that

$$V_1(\delta) = V_2(\delta) = \gamma. \tag{4.32}$$

Then we find

$$\left.\begin{aligned} Z_1 &= 2(\gamma/\delta), \\ Z_2 &= 2\left(\frac{1-\delta\gamma}{1-\delta^2}\right). \end{aligned}\right\} \tag{4.33}$$

As mentioned above we only treat the case $Z_1 \geqslant 0$ and $Z_2 \geqslant 0$, that is $0 \leqslant \gamma \leqslant 1/\delta$. The total amount of vorticity is equal to the circulation Γ, namely:

$$\Gamma = 2\pi \int_0^\infty Z(r) r\, dr = 2\pi \tag{4.34}$$

that is, a constant. The vorticity and the velocity distribution is shown in figure 5.

From (2·7) we see that for $k = 0$ inside each region, $F \equiv 0$ because of $Z' \equiv 0$. Then the disturbance equation (2.5) has the simple form

$$\phi'' + \frac{1}{r}\phi' - \frac{m^2}{r^2}\phi = 0, \tag{4.35}$$

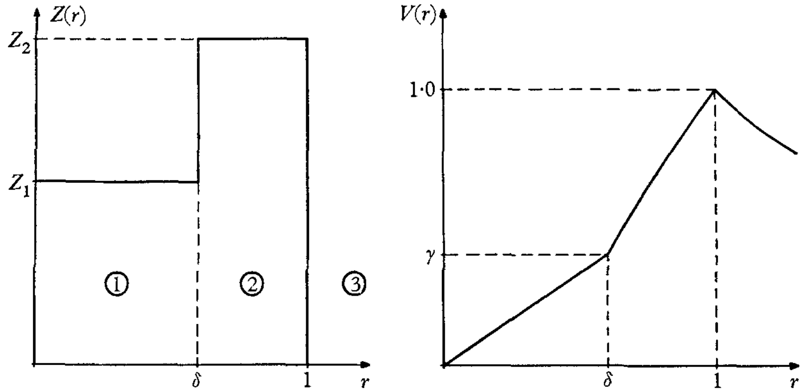

FIGURE 5. The vorticity and velocity distribution due to equations (4.29) and (4.30) respectively.

the general solution of which is for $m \neq 0$

$$\phi(r) = A^* r^m + B^* r^{-m}. \tag{4.36}$$

We shall only treat this case $k = 0$ in the following.

The boundary conditions at $r = 0$ and $r = \infty$ require solutions for each region as follows

$$\left.\begin{aligned} \phi_1(r) &= A_1^* r^m, \\ \phi_2(r) &= A_2^* r^m + B_2^* r^{-m}, \\ \phi_3(r) &= \qquad\quad B_3^* r^{-m}, \end{aligned}\right\} \tag{4.37}$$

with $m \geqslant 1$. At the boundaries of the regions the radial velocity u_r and the pressure p must be steady. This means that by (2.4) and (2.10)

$$\left.\begin{aligned} &\phi = \text{const.} \\ &[V - (\beta/m)\, r]\, \phi' - Z\phi = \text{const.} \end{aligned}\right\} \tag{4.38}$$

on each side of $r = 1$ and $r = \delta$. From (4.37) and (4.38) we find at $r = 1$:

$$[2(m-\beta) - Z_2]\, A_2^* - Z_2 B_2^* = 0, \tag{4.39}$$

while at $r = \delta$ $\quad [Z_2 - Z_1]\, \delta^m A_2^* + [mZ_1 - 2\beta + Z_2 - Z_1]\, \delta^{-m} B_2^* = 0. \qquad (4.40)$

For non-trivial solutions the determinant of the linear equations (4.39) and (4.40) for A_2^* and B_2^* must vanish. Thus a quadratic equation for the eigenvalue β is obtained, the solution of which is

$$2\beta = m + (m-1)\,\tfrac{1}{2}Z_1 \pm ([m - Z_2 - (m-1)\,\tfrac{1}{2}Z_1]^2 - Z_2(Z_2 - Z_1)\,\delta^{2m})^{\frac{1}{2}}. \tag{4.41}$$

The flow is unstable, if β becomes complex. This is only possible, if

$$Z_2 > Z_1 \geqslant 0, \quad \text{that is} \quad \delta > \gamma \geqslant 0. \tag{4.42}$$

This condition corresponds to the conditions of an extremum value of the vorticity for $r > 0$ as shown in §3. Therefore the Rankine vortex which is present for $Z_2 = 0$ or $Z_2 = Z_1$ is always stable as mentioned above. If (4.42) is satisfied, the cyclic frequency is

$$\beta_r = \tfrac{1}{2}[m+(m-1)\tfrac{1}{2}Z_1] \tag{4.43}$$

and the growth rate is

$$\beta_i = \tfrac{1}{2}(Z_2(Z_2-Z_1)\,\delta^{2m} - [m-Z_2-(m-1)\tfrac{1}{2}Z_1]^2)^{\frac{1}{2}}. \tag{4.44}$$

Inserting (4.33) into (4.43) and (4.44), β_r and β_i are obtained as functions of δ, γ and m, namely

$$\beta_r = \frac{1}{2}\left[m+(m-1)\frac{\gamma}{\delta}\right], \tag{4.45}$$

$$\beta_i = \frac{(a_0+2a_1\gamma+a_2\gamma^2)^{\frac{1}{2}}}{2(1-\delta^2)\,\delta}, \tag{4.46}$$

where
$$\left.\begin{aligned} a_0 &= \delta^2\{4\delta^{2m} - [m(1-\delta^2)-2]^2\}, \\ a_1 &= -\delta\{2(1+\delta^2)\,\delta^{2m} + [m(1-\delta^2)-2]\,[1+\delta^2-m(1-\delta^2)]\}, \\ a_2 &= 4\delta^{2m+2} - [1+\delta^2-m(1-\delta^2)]^2. \end{aligned}\right\} \tag{4.47}$$

Let us first look for which pairs of the profile parameters (δ, γ) the flow becomes unstable. To do this we calculate the neutral curves in the (δ, γ)-plane which are given by the equation

$$a_2\gamma^2 + 2a_1\gamma + a_0 = 0. \tag{4.48}$$

It is found that for $m = 1$ and $m = 2$ the flow is stable if $0 < \delta < 1$ and $0 \leqslant \gamma \leqslant \delta$. For $m \geqslant 3$ we have instability for all values of (δ, γ) lying inside the neutral curves which are shown in figure 6 for $m = 3; 4; 5$. It is evident that the presence

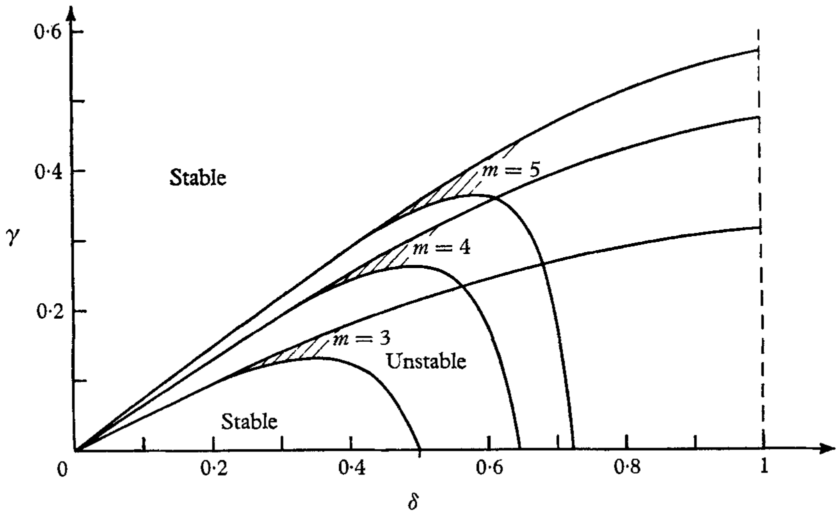

FIGURE 6. The neutral curves in the (δ, γ)-plane for $m = 3, 4, 5$.

of the inner vorticity $Z_1 \sim \gamma$ has not always a stabilizing influence as one might have expected. For example, at $m = 3$ and $\delta = 0{\cdot}5$ the flow is stable for $\gamma = 0$, but unstable for $0 < \gamma < \frac{3}{14}$.

The eigenvalues for amplified disturbances as function of m can be calculated from the equations (4.45) and (4.46) for fixed profile parameters (δ, γ); the cyclic frequency β_r depends linearly on m by (4.45). The growth rates β_i are shown in figure 7 for $\gamma = 0$ and various values of δ. In this case only a sheet of vorticity with magnitude $Z_2 = 2[1-\delta^2]^{-1}$ and thickness $d = 1-\delta$ is present, while the total amount of the vorticity, which is equal to the circulation Γ, is constant by (4.34). We see on figure 7 that with increasing δ the maximum amplification

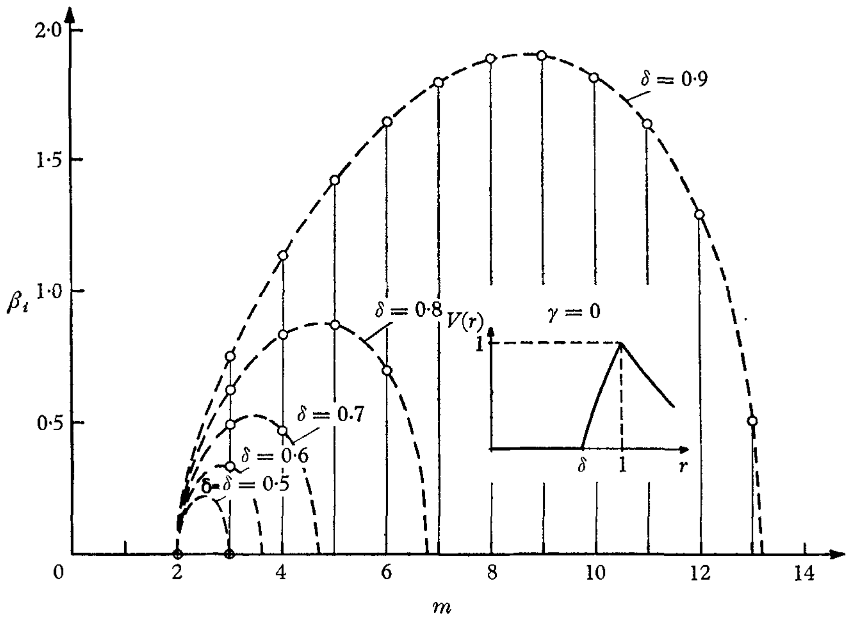

FIGURE 7. The growth rate β_i *vs.* m for $\gamma = 0$ and $\delta = 0{\cdot}9, 0{\cdot}8, 0{\cdot}7, 0{\cdot}6, 0{\cdot}5$.

increases strongly as well as the unstable range of m, that is the thinner the sheet in which the vorticity is distributed the more unstable it becomes. This is analogous to the plane linear shear layer (cf. Michalke & Schade 1963). Also here the result tends to that of the linear shear layer, if the mean radius of the vorticity sheet tends to infinity.

In figures 8 and 9 the growth rate β_i is shown for $\gamma = 0{\cdot}1$ and $\gamma = 0{\cdot}2$ respectively, that is for $Z_1 \neq 0$. It is obvious that with increasing γ the maximum growth rate as well as the unstable range of m slightly decreases. For $\delta \to 1$ the vorticity distribution Z_2 tends to a cylindrical vortex sheet. But, although the eigenvalues tend to finite limiting values, they do not agree with those obtained from a stability calculation using directly a vortex sheet (see §4.1). Thus the limiting process $\delta \to 1$ in (4.45) and (4.46) is not applicable. This may be due to the use of the first condition of (4.38) at the boundaries $r = \delta$ and $r = 1$ instead of the corresponding condition (4.15).

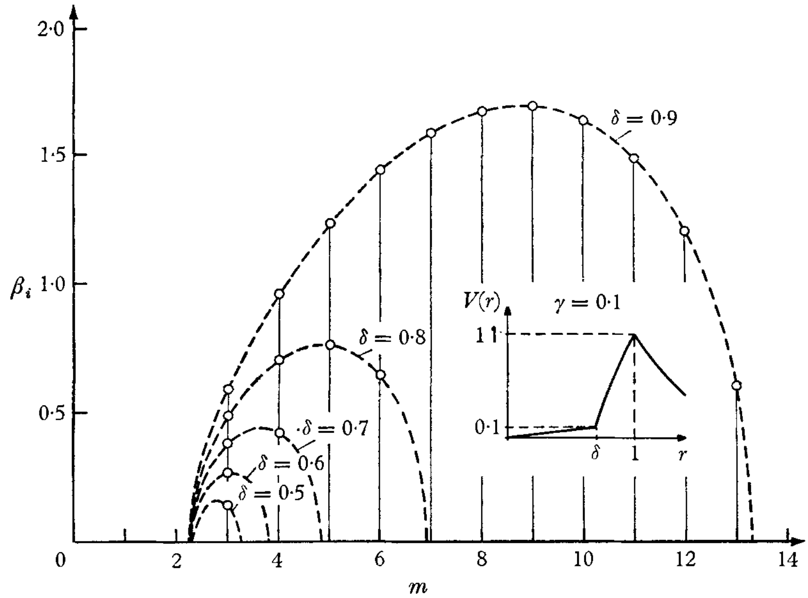

FIGURE 8. The growth rate β_i *vs.* m for $\gamma = 0{\cdot}1$ and $\delta = 0{\cdot}9, 0{\cdot}8, 0{\cdot}7, 0{\cdot}6, 0{\cdot}5$.

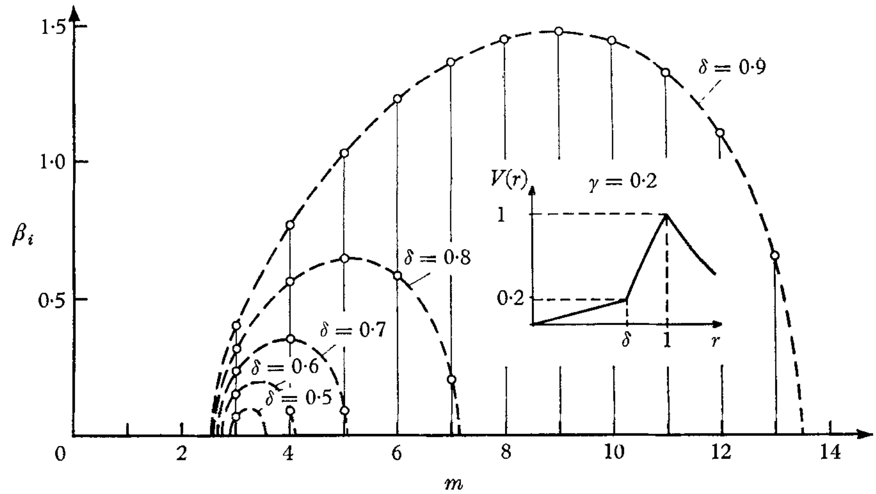

FIGURE 9. The growth rate β_i *vs.* m for $\gamma = 0{\cdot}2$ and $\delta = 0{\cdot}9, 0{\cdot}8, 0{\cdot}7, 0{\cdot}6, 0{\cdot}5$.

4.3. *The instability of a vortex with a special continuous vorticity distribution.*

While in §§4.1 and 4.2 the instability of vortex-type flows was investigated for which the disturbance equation (2.5) became simple and could be solved in closed form, we here assume a vortex with continuously distributed vorticity. Thus the eigenvalue of (2.15) must be computed numerically. The velocity distribution chosen is given by

$$V(r) = r^{-1}\left[1 - \exp\left(-\frac{n}{n+1}\, r^{2(n+1)}\right)\right] \tag{4.49}$$

and the corresponding vorticity distribution is

$$Z(r) = 2n\,r^{2n}\exp\left(-\frac{n}{n+1}r^{2(n+1)}\right), \tag{4.50}$$

where n is a profile parameter. The vorticity distribution has a maximum at $r = 1$, and for increasing profile parameter n the vorticity is increasingly concentrated at the cylindrical sheet at $r = 1$. In the limit $n \to \infty$ the flow tends to that of a cylindrical vortex sheet at $r = 1$ which was investigated in §4.1. The circulation Γ_∞ of the flow at infinity

$$\Gamma_\infty = \lim_{r\to\infty}\Gamma(r) = 2\pi \tag{4.51}$$

is constant. Furthermore, for all r we have $VZ \geqslant 0$.

In the following we restrict ourselves to the case $n = 2$. Then we have

$$\left.\begin{aligned} V(r) &= r^{-1}[1-\exp(-2r^6/3)], \\ Z(r) &= 4r^4\exp(-2r^6/3), \\ Z'(r) &= 16r^3[1-r^6]\exp(-2r^6/3). \end{aligned}\right\} \tag{4.52}$$

The velocity and the vorticity distribution of this flow is shown in figure 10.

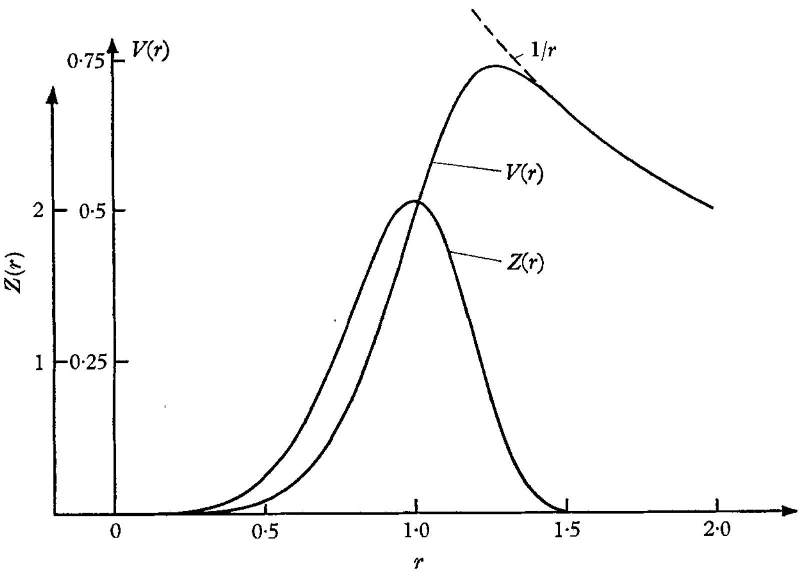

FIGURE 10. The vorticity and velocity distribution given by (4.51).

In order to evaluate the eigenvalues β for amplified disturbances, the disturbance equation (2.5)

$$\left[\frac{r}{m^2+k^2r^2}\phi'\right]' - \left[\frac{1}{r}+F\right]\phi = 0 \tag{4.53}$$

has to be solved with boundary conditions

$$\phi(0) = 0, \tag{4.54}$$

$$\lim_{r\to\infty}\phi(r)/r = 0. \tag{4.55}$$

The function $F(r)$ can be written by (2.7) for $m \neq 0$

$$F(r) = m^{-2}\left[\frac{V}{r}-\frac{\beta}{m}\right]^{-1}\cdot\left[1+\left(\frac{kr}{m}\right)^2\right]^{-1}\left\{Z'-\left(\frac{k}{m}\right)^2 2rZ \right.$$
$$\left.\times\left[\left[1+\left(\frac{kr}{m}\right)^2\right]^{-1}+\frac{V}{r}\left[\frac{V}{r}-\frac{\beta}{m}\right]^{-1}\right]\right\}. \quad (4.56)$$

It is evident from (4.56) and (4.52) that $F(r)$ is antisymmetric in r. Furthermore, for $\beta_i \neq 0$

$$F(0) = 0 \quad (4.57)$$

and

$$\lim_{r\to\infty} F(r) = 0. \quad (4.58)$$

From the latter it follows that the disturbance equation (4·53) tends asymptotic-ally to the equation (4.4). Thus the asymptotic solution satisfying (4.55) has to be for $r \to \infty$.

$$\phi(r) \sim k^m r \frac{dK_m(kr)}{dr}. \quad (4.59)$$

For $\beta_i \neq 0$ and $r \to 0$ it follows from (4.57) that (4.53) tends to (4.35). Thus by (4.54) for $r \to 0$

$$\phi(r) \sim r^m. \quad (4.60)$$

The order of the differential equation (4.53) can be reduced, if we substitute

$$\phi(r) = \exp\int\frac{m^2+k^2r^2}{r}\,\Phi\,dr \quad (4.61)$$

into (4.53). Then we obtain the corresponding Riccati equation in Φ

$$\Phi' = \frac{1}{r}\left[1-(m^2+k^2r^2)\,\Phi^2\right]+F(r), \quad (4.62)$$

where $F(r)$ is defined by (4.56) and (4.52). The inversion of (4.61) gives

$$\Phi(r) = \frac{r}{m^2+k^2r^2}\,\frac{\phi'}{\phi}. \quad (4.63)$$

Hence by (4.60) we have for $r = 0$

$$\Phi(0) = 1/m \quad (4.64)$$

and for $r \to \infty$ by (4.59) $\quad \Phi(r) = -\left[m+kr\frac{K_{m-1}(kr)}{K_m(kr)}\right]^{-1}, \quad (4.65)$

where the differential equation and the recurrence relations for the modified Bessel functions have been used. Thus $\Phi(\infty) = 0$ for $k \neq 0$ and $\Phi(\infty) = -m^{-1}$ for $k = 0$.

The method of computing the eigenvalues β of the disturbance equation (4.53) was similar to that described by Michalke (1965*b*). For fixed values m and k the complex differential equation (4.62) was integrated numerically starting from $r = 0$ to $r = r_0 = 1{\cdot}1$ which gives $\Phi_1(r_0)$ and backwards from $r = 2$ to $r = r_0$ which gives $\Phi_2(r_0)$. At $r = 2$ it was found $|F(2)| < 10^{-10}$ for $m \geqslant 2$ so that the value of $\Phi(2)$ can be calculated from the asymptotic representation of $\Phi(r)$ given in (4.65). At $r = 0$ the derivative $\Phi'(0)$ was found to be zero. For arbitrarily chosen different pairs of $\beta = \beta_r + i\beta_i$ the difference

$$G(\beta_r, \beta_i) = \Phi_1(r_0) - \Phi_2(r_0) \quad (4.66)$$

was evaluated and improved values of β were calculated from the approximate zeros of $G(\beta_r, \beta_i)$ by linear interpolation. This procedure was then repeated until

$|G|$ was small enough. The computation was performed on a Zuse Z 23 v digital computer using a Runge–Kutta procedure.

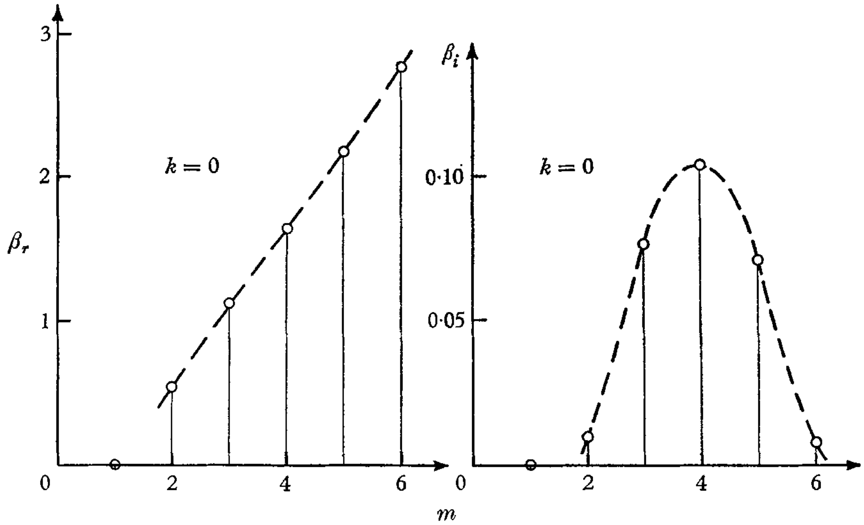

FIGURE 11. The frequency β_r and the growth rate β_i as functions of m for $k = 0$.

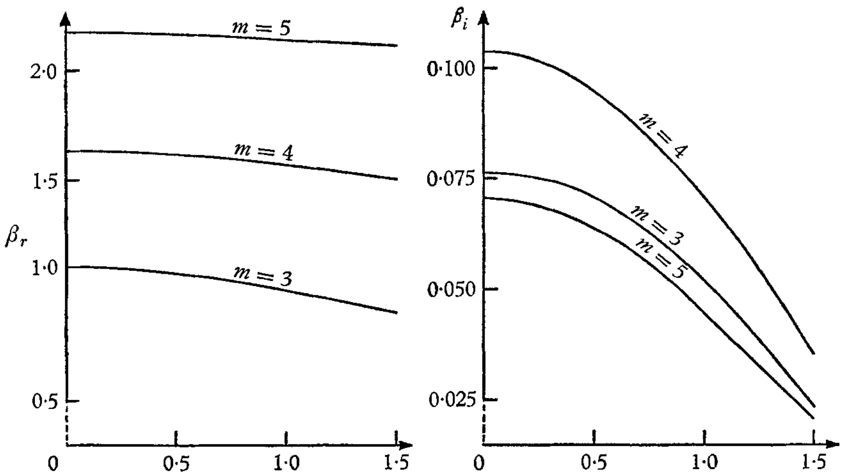

FIGURE 12. The frequency β_r and the growth rate β_i as functions of k for m = 3, 4, 5.

For $k = 0$ the eigenvalues $\beta_r(m)$ and $\beta_i(m)$ are shown in figure 11 for $m = 2$; 3; 4; 5; 6, although the eigenvalues are not mathematically restricted to integral m. Maximum amplification is found at $m = 4$. The neutral value for $m = 1$ is also included in figure 11. The other neutral eigenvalue with $m > 6$ cannot be computed in this way, because apparently $F(r)$ becomes singular at two points where $(V/r)_{r_c} = \beta_r/m$. The computation has also been performed for three-dimensional disturbances with $k \neq 0$. In figure 12 the eigenvalues are shown for $m = 3$; 4; 5 as function of k. We see that for $k \neq 0$ the growth rate β_i becomes smaller than for $k = 0$. Therefore also here the three-dimensional disturbances are less unstable than the two-dimensional disturbances as suggested by (3.12). But it is evident that the vortex-type flow given by the vorticity distribution (4.52) is unstable although the Rayleigh criterion predicts stability.

5. Conclusion

In the introduction it was shown what vorticity distribution in a vortex occurring in two-dimensional free boundary layers must be expected. We found that the vorticity should be arranged in a nearly circular band somewhere within which it should reach its maximum value. A vorticity distribution with a maximum outside of the vortex axis was also found for the vortices in a Kármán vortex street by Timme (1957). The question to be answered was when such an arrangement of vorticity might become unstable and so initiate the vortex breakdown. To simplify the problem we assumed the vorticity distribution to be approximately axisymmetric and, by neglecting the influence of viscosity, to be stationary. By this means the inviscid stability theory of inviscid rotating fluid was applicable. It was found that a vortex flow consisting of vorticity of one sign only can be unstable to cylinder-symmetrical disturbances, if the vorticity has an extremal value outside the axis.

Thus the statement of Howard & Gupta (1962) that the Rayleigh criterion is not applicable for non-axisymmetric disturbances was confirmed. This type of instability has been observed experimentally by Weske & Rankin (1963) in a special arrangement. Furthermore, it was found that for the special cases investigated here the two-dimensional cylinder-symmetric disturbances are apparently more strongly amplified than the corresponding three-dimensional ones.

On the basis of the results of the vortex models used here we can expect that vortices in a two-dimensional free boundary layer may also become unstable, although their more complicated shape may influence the instability properties. But we cannot conclude that this type of vortex instability is responsible for vortex breakdown, since so far no experimental confirmation has been possible. Thus an experimental investigation of the vortex breakdown phenomenon in two-dimensional free boundary layers seems to be very desirable though it may be very difficult to handle.

Another question remaining open is what may occur if a peripheral disturbance with non-integral m is excited in a vortex. This could be arranged, for instance, by means of a ribbon parallel to the z-axis oscillating in the radial direction, although the frequency β then must be real and m complex. It is generally agreed that cylinder-symmetric disturbances with integral m's are physically reasonable. Nevertheless, the disturbance equation possesses also solutions for non-integral m's, but then the disturbed flow is non-unique. But it should be worthwhile to investigate whether just these solutions may cause the vortex breakdown leading to turbulence.

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